

NONCOMMUTATIVE RIGIDITY OF THE MODULI STACK OF STABLE POINTED CURVES

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ABSTRACT. We prove that the second Hochschild cohomology of $\overline{\mathcal{M}}_{g,n}$, the moduli stack of stable n -pointed genus g curves, is trivial except when $(g, n) = (0, 5)$.

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1. INTRODUCTION

It is proven in [Hac08] that the moduli stack of stable n -pointed genus g curves, which is denoted by $\overline{\mathcal{M}}_{g,n}$, is rigid over any field \mathbf{k} of characteristic zero. The case $g = 0$ over fields of positive characteristics is also treated in [FM14].

In this paper we investigate the rigidity of $\overline{\mathcal{M}}_{g,n}$ in a stronger sense. Namely, we consider the deformations of the \mathbf{k} -linear abelian category $\mathrm{Coh} \overline{\mathcal{M}}_{g,n}$ (we call them *noncommutative deformations* as well). Also, over the field of complex numbers \mathbb{C} , we consider the associated analytic stack $\overline{\mathcal{M}}_{g,n}^{\mathrm{an}}$ and think of its rigidity in the sense of generalized complex geometry of Hitchin [Hit03].

The notion of deformations of abelian categories was systematically treated in [LVdB06]. In [LVdB05], it was shown that the deformation theory of abelian categories is described by means of the *Hochschild cohomology* of the category. For a smooth proper Deligne-Mumford stack X over \mathbf{k} , it is defined as

$$\mathrm{HH}^i(X) = \mathrm{HH}_{\mathbf{k}}^i(X) := \mathrm{HH}_{\mathbf{k}}^i(\mathrm{Coh} X) \cong \mathrm{Ext}_{X \times_{\mathbf{k}} X}^i(\mathcal{O}_{\Delta}, \mathcal{O}_{\Delta}), \quad (1.1)$$

where Δ is the image of the diagonal morphism $X \hookrightarrow X \times_{\mathbf{k}} X$. The last term of (1.1) is more explicitly described, via the *Hochschild-Kostant-Rosenberg isomorphism*, as

$$\mathrm{HH}_{\mathbf{k}}^i(X) \cong \bigoplus_{p+q=i} H^q(X, \wedge^p \Theta_{X/\mathbf{k}}). \quad (1.2)$$

Here $\Theta_{X/\mathbf{k}} = \Theta_X$ is the locally free sheaf of algebraic vector fields on X over \mathbf{k} .

According to [LVdB05], the second Hochschild cohomology

$$\mathrm{HH}^2(X) \cong H^0(\wedge^2 \Theta_X) \oplus H^1(\Theta_X) \oplus H^2(\mathcal{O}_X) \quad (1.3)$$

classifies the first order deformations of the category $\mathrm{Coh} X$. The second term of the RHS corresponds to the deformations of X in the usual sense. From the point of view of the generalized complex geometry, the RHS of (1.3) classifies the isomorphism classes of the first order deformations of the generalized complex structure on X^{an} [Gua11, Section 5.3].

Therefore, if the second Hochschild cohomology is trivial, the stack is rigid in both senses. In this paper we show that the triviality holds for $\overline{\mathcal{M}}_{g,n}$ except one case:

Theorem 1.1 (In characteristic zero). *For any (g, n) , the second Hochschild cohomology $\mathrm{HH}^2(\overline{\mathcal{M}}_{g,n})$ is isomorphic to $H^0(\overline{\mathcal{M}}_{g,n}, \wedge^2 \Theta_{\overline{\mathcal{M}}_{g,n}})$ through the HKR isomorphism (1.3). It is of dimension 6 for $(g, n) = (0, 5)$, and trivial otherwise.*

Since our proof of Theorem 1.1 relies on the Kodaira type vanishing theorems, we exclusively work over a field of characteristic zero. The second term of the RHS of (1.3) is known to be trivial by [Hac08, Proposition 4.1]. Hence the first claim of Theorem 1.1 reduces to the vanishing of the third term of (1.3), which will be proven in Theorem 3.1 of this paper. In fact it also follows from a stronger result [AC98, Theorem 2.2] modulo the Hodge decomposition for smooth proper Deligne-Mumford stacks (Lemma 2.8). Our proof can be seen as a shortcut version which is enough for what we want. The rest of Theorem 1.1 is a combination of Theorem 3.2 (the case $\dim \overline{\mathcal{M}}_{g,n} \neq 2$), Lemma 3.7 (the case $(g, n) = (0, 5)$), and Proposition 3.9 (the case $(g, n) = (1, 2)$).

The proof of Theorem 3.2 is based on the inductive structure of the moduli stack $\overline{\mathcal{M}}_{g,n}$ (Lemma 2.13), ampleness of the log canonical bundle $\omega_{\overline{\mathcal{M}}_{g,n}}(\mathcal{B})$ (Proposition 2.12), and the positivity of the ψ -classes (Lemma 2.11). For the sake of completeness of the proof, in Section 2.1, we generalize the notion of orientation sheaf and some vanishing theorems of [DI87] to normal crossing pairs of Deligne-Mumford stacks after [Sat12]. This would be of independent interest.

The proof of Theorem 3.1 is based again on the inductive structure, results established in Section 2.1, and a vanishing theorem due to Harer on the rational homology groups of the open substack $\mathcal{M}_{g,n}$ (Fact 3.6).

Recall that the rigidity of $\overline{\mathcal{M}}_{g,n}$ in the usual sense is a first nontrivial case of the Geometric syzygy principle due to Kapranov [Kap] (see also [Hac08, Introduction]). We are not sure if Theorem 1.1 can be understood in this context.

Lastly we show in Lemma 3.8 the triviality of $\mathrm{HH}^3(\overline{\mathcal{M}}_{0,5})$, and hence the unobstructedness of the deformations of $\mathrm{Coh} \overline{\mathcal{M}}_{0,5}$ (see [LVdB05, Theorem 3.1]). Since $\mathrm{HH}^1(\overline{\mathcal{M}}_{0,5})$ is easily seen to be trivial, all these calculations tell us that there should be a six dimensional non-singular formal moduli of deformations of the category $\mathrm{Coh} \overline{\mathcal{M}}_{0,5}$. As a matter of fact, we can construct *global* (i.e. not merely formal) family of such categories as blowups of noncommutative projective planes in four points after [VdB01]. In this direction, it would be interesting to think of the compactified moduli spaces of such categories using the methods of [AOU14]. From the point of view of the generalized complex geometry, our calculations imply that the holomorphic bivectors on $\overline{\mathcal{M}}_{0,5}$ are automatically Poisson so as to yield a six dimensional family of holomorphic Poisson deformations of $\overline{\mathcal{M}}_{0,5}^{\mathrm{an}}$ (see [Gua11, Section 5.3]).

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Notations and conventions. Throughout the paper, we work over an arbitrary field \mathbf{k} . Its characteristic is assumed to be zero unless otherwise stated.

2. PRELIMINARIES

2.1. Logarithmic differential forms on Deligne-Mumford stacks. For a variety X over \mathbf{k} , the p -th exterior product of the sheaf of Kähler differentials will be denoted by Ω_X^p . Its double dual $(\Omega_X^p)^{**}$ will be denoted by $\tilde{\Omega}_X^p$.

Lemma 2.1. *Let Y be a smooth affine variety and G a finite group acting on Y . Let $X := Y/G$ be the quotient by the G -action and $\pi: Y \rightarrow X$ the quotient morphism. Then we have a canonical isomorphism*

$$\tilde{\Omega}_X^p \xrightarrow{\sim} (\pi_* \Omega_Y^p)^G,$$

where the RHS is the G -invariant part of the coherent sheaf $\pi_* \Omega_Y^p$.

Proof. This is a special case of [Bri98, Theorem 2]. Note that since G is finite, being horizontal (see [Bri98, Introduction] for the definition) is an empty condition and π contracts no divisor. \square

Corollary 2.2. *Let \mathcal{X} be a smooth Deligne-Mumford stack separated over \mathbf{k} which admits a coarse moduli scheme $c: \mathcal{X} \rightarrow X$. Then there exist a natural isomorphism*

$$\tilde{\Omega}_X^p \xrightarrow{\sim} \mathbb{R}c_* \Omega_{\mathcal{X}}^p. \quad (2.1)$$

Proof. The natural morphism $c_* \Omega_{\mathcal{X}}^p \rightarrow \mathbb{R}c_* \Omega_{\mathcal{X}}^p$ is an isomorphism by [AV02, Lemma 2.3.4]. In the rest we show that the canonical morphism $c^*: \tilde{\Omega}_X^p \rightarrow c_* \Omega_{\mathcal{X}}^p$ is an isomorphism.

By [AV02, Lemma 2.2.3], étale locally on X , the morphism c is of the form

$$c: [U/G] \rightarrow U/G. \quad (2.2)$$

Here G is a finite group acting algebraically on a smooth affine variety U . Recall that there exists a canonical equivalence of categories $\mathrm{Coh}[U/G] \xrightarrow{\sim} \mathrm{Coh}^G U$ and that, under this equivalence, the pushforward functor $c_*: \mathrm{Coh}[U/G] \rightarrow \mathrm{Coh} U/G$ is identified with

$$\mathrm{Coh}^G U \rightarrow \mathrm{Coh} U/G; \quad F \mapsto (\pi_* F)^G. \quad (2.3)$$

Here $\pi: U \rightarrow U/G$ is the quotient morphism. Therefore, by applying Lemma 2.1 to $F = \Omega_U^p$, we obtain the conclusion. \square

Lemma 2.3. *Let \mathcal{X} be a smooth proper Deligne-Mumford stack over \mathbb{C} which admits a projective coarse moduli scheme. Then the Hodge symmetry*

$$\dim H^q(\mathcal{X}, \Omega_{\mathcal{X}}^p) = \dim H^p(\mathcal{X}, \Omega_{\mathcal{X}}^q) \quad (2.4)$$

holds.

Proof. Let $c: \mathcal{X} \rightarrow X$ be the morphism to the coarse moduli scheme. Since X is a projective V -manifold, the Hodge symmetry

$$\dim H^q(X, \tilde{\Omega}_X^p) = \dim H^p(X, \tilde{\Omega}_X^q) \quad (2.5)$$

holds for any (p, q) by [Ste77, (1.12) Theorem]. By Corollary 2.2 we obtain the isomorphisms

$$H^p(X, \tilde{\Omega}_X^q) \cong H^p(X, \mathbb{R}_{c*} \Omega_{\mathcal{X}}^q) \cong H^p(\mathcal{X}, \Omega_{\mathcal{X}}^q), \quad (2.6)$$

to conclude the proof. \square

Next we define the notion of orientation sheaf for normal crossing pairs of Deligne-Mumford stacks. In the case of varieties, this is originally introduced in [Del71, 3.1.4].

Definition-Proposition 2.4. Let $(\mathcal{X}, \mathcal{D})$ be a normal crossing pair of a smooth Deligne-Mumford stack \mathcal{X} and a normal crossing divisor \mathcal{D} on it. For each non-negative integer k , let $\mathcal{D}^{(k)} \subset \mathcal{X}$ be the closed substack defined by the following rule;

- $\mathcal{D}^{(0)} = \mathcal{X}$.
- $\mathcal{D}^{(1)} = \mathcal{D}$.
- $\mathcal{D}^{(k)} = \text{Sing } \mathcal{D}^{(k-1)}$ for $k > 1$.

Let $\varphi^k: \mathcal{D}^k \rightarrow \mathcal{D}^{(k)}$ be the normalization. Take an atlas $U \rightarrow \mathcal{X}$ such that $D := \mathcal{D} \times_{\mathcal{X}} U$ is a SNC divisor. Set $D^{(k)} := \mathcal{D}^{(k)} \times_{\mathcal{X}} U$, $D^k := \mathcal{D}^k \times_{\mathcal{X}} U$ and $\varphi_U^k: D^k \rightarrow U$ the natural morphism. Consider a locally constant sheaf E_{D^k} on D^k of finite sets of order k defined as follows: for a connected open set $W \subset D^k$, let $E_{D^k}(W)$ be the set of irreducible components of D which contain W .

Let $p_i: U \times_{\mathcal{X}} U \rightarrow U$ and $p_i^k: D^k \times_{\mathcal{D}^k} D^k \rightarrow D^k$ be the projections for $i = 1, 2$. We define the gluing isomorphisms $\phi_{12}^U: (p_2^k)^* E_{D^k} \xrightarrow{\sim} (p_1^k)^* E_{D^k}$ as follows. Let $E_{D^k \times_{\mathcal{D}^k} D^k}$ be the local system on $D^k \times_{\mathcal{D}^k} D^k$ corresponding to the atlas $U \times_{\mathcal{X}} U \rightarrow \mathcal{X}$. Since $p_i^* D$ is canonically isomorphic to $D \times_{\mathcal{D}} D$, we can define natural isomorphisms $b_i: E_{D^k \times_{\mathcal{D}^k} D^k} \xrightarrow{\sim} (p_i^k)^* E_{D^k}$ for $i = 1, 2$. Then we set

$$\phi_{12}^U := b_1 \circ b_2^{-1}: (p_2^k)^* E_{D^k} \xrightarrow{\sim} (p_1^k)^* E_{D^k}.$$

On a triple product $D^k \times_{\mathcal{D}^k} D^k \times_{\mathcal{D}^k} D^k$ we can check the cocycle condition

$$p_{12}^* \phi_{12}^U \circ p_{23}^* \phi_{23}^U = p_{13}^* \phi_{13}^U,$$

where $p_{ij}: D^k \times_{\mathcal{D}^k} D^k \times_{\mathcal{D}^k} D^k \rightarrow D^k \times_{\mathcal{D}^k} D^k$ is the projections for $1 \leq i < j \leq 3$. Thus these data determines a local system $E_{\mathcal{D}^k}$ on \mathcal{D}^k .

The local system $E_{\mathcal{D}^k}$ induces a locally free sheaf $\mathcal{F}_{\mathcal{D}^k} := \mathcal{O}_{\mathcal{D}^k}^{\oplus E_{\mathcal{D}^k}}$ of rank k on \mathcal{D}^k . That is, for a connected open subset $W \subset D^k$ we set

$$\mathcal{F}_{D^k}(W) = \bigoplus_{\Delta \in E_{D^k}(W)} \mathcal{O}_{D^k}(W) \cdot e_{\Delta},$$

where e_{Δ} is a formal element corresponding to the irreducible component Δ . The gluing isomorphisms ϕ_{12}^U of E_{D^k} provides us with the canonical isomorphisms $p_2^* \mathcal{F}_{D^k} \xrightarrow{\sim} p_1^* \mathcal{F}_{D^k}$ satisfying the cocycle conditions.

Finally we define the *orientation sheaf* as the line bundle $\mathcal{E}_{\mathcal{D}^k} := \det \mathcal{F}_{\mathcal{D}^k}$ on \mathcal{D}^k . One can easily check $\mathcal{E}_{\mathcal{D}^k}^{\otimes 2} \simeq \mathcal{O}_{\mathcal{D}^k}$. Since the local system $E_{\mathcal{D}^1}$ is trivial, we have $\mathcal{E}_{\mathcal{D}^1} \simeq \mathcal{O}_{\mathcal{D}^1}$.

By using the line bundles $\mathcal{E}_{\mathcal{D}^k}$, we obtain the following exact sequence.

Lemma 2.5. *Let $(\mathcal{X}, \mathcal{D})$ be a normal crossing pair of Deligne-Mumford stacks. Then for each $p \geq 0$ there exist the following mutually dual long exact sequences.*

$$0 \rightarrow \Omega_{\mathcal{X}}^p(\log \mathcal{D})(-\mathcal{D}) \rightarrow \Omega_{\mathcal{X}}^p \rightarrow \varphi_*^1 \Omega_{\mathcal{D}^1}^p(\mathcal{E}_{\mathcal{D}^1}) \xrightarrow{r_1} \varphi_*^2 \Omega_{\mathcal{D}^2}^p(\mathcal{E}_{\mathcal{D}^2}) \xrightarrow{r_2} \cdots \quad (2.7)$$

$$\begin{aligned} 0 \rightarrow \Theta_{\mathcal{X}}^p(-\log \mathcal{D}) \rightarrow \Theta_{\mathcal{X}}^p \rightarrow \varphi_*^1 (\Theta_{\mathcal{D}^1}^{p-1}(\mathcal{E}_{\mathcal{D}^1}) \otimes \det(\mathcal{N}_{\mathcal{D}^1/\mathcal{X}})) \\ \rightarrow \varphi_*^2 (\Theta_{\mathcal{D}^2}^{p-2}(\mathcal{E}_{\mathcal{D}^2}) \otimes \det(\mathcal{N}_{\mathcal{D}^2/\mathcal{X}})) \rightarrow \varphi_*^3 (\Theta_{\mathcal{D}^3}^{p-3}(\mathcal{E}_{\mathcal{D}^3}) \otimes \det(\mathcal{N}_{\mathcal{D}^3/\mathcal{X}})) \rightarrow \cdots \end{aligned} \quad (2.8)$$

Proof. For each integer k , the restriction homomorphism

$$r_k: \varphi_*^k \Omega_{\mathcal{D}^k}^p(\mathcal{E}_{\mathcal{D}^k}) \rightarrow \varphi_*^{k+1} \Omega_{\mathcal{D}^{k+1}}^p(\mathcal{E}_{\mathcal{D}^{k+1}})$$

is defined as follows: Choose an atlas $U \rightarrow \mathcal{X}$ on which $D := \mathcal{D} \times_{\mathcal{X}} U$ is a SNC divisor. Denote by $D = \bigcup_{i \in I} D_i$ the irreducible decomposition of D . Then $D^k := \mathcal{D}^k \times_{\mathcal{X}} U$ can be written as

$$D^k = \coprod_{i_1 < \cdots < i_k} D_{i_1 \dots i_k},$$

where $D_{i_1 \dots i_k} := D_{i_1} \cap \cdots \cap D_{i_k}$. Let $\lambda_{j,k}: D_{i_1 \dots i_{k+1}} \rightarrow D_{i_1 \dots i_{j-1} i_{j+1} \dots i_{k+1}}$ be the natural closed immersion. Then we define

$$r_k^U: (\varphi_U^k)_* \Omega_{D^k}^i(\mathcal{E}_{D^k}) \rightarrow (\varphi_U^{k+1})_* \Omega_{D^{k+1}}^i(\mathcal{E}_{D^{k+1}})$$

as the sum over $j = 1, 2, \dots, k+1$ of the homomorphisms

$$\lambda_{j,k}^* \otimes (e_{D_{i_j}} \wedge): \lambda_{j,k}^* \Omega_{D_{i_1 \dots i_{j-1} i_{j+1} \dots i_{k+1}}}^i(\mathcal{E}_{D_{i_1 \dots i_{j-1} i_{j+1} \dots i_{k+1}}}) \rightarrow \Omega_{D_{i_1 \dots i_{k+1}}}^i(\mathcal{E}_{D_{i_1 \dots i_{k+1}}}),$$

where $e_{D_{i_j}} \in \mathcal{F}_{D^k}$ is the element corresponding to the irreducible component D_{i_j} of D .

We can check that $p_1^* r_k^U = p_2^* r_k^U$ holds for the projections $p_i: U \times_{\mathcal{X}} U \rightarrow U$ by the construction of $\mathcal{E}_{\mathcal{D}^k}$. Hence we see that the homomorphism $r_k: \varphi_*^k \Omega_{\mathcal{D}^k}^p(\mathcal{E}_{\mathcal{D}^k}) \rightarrow \varphi_*^{k+1} \Omega_{\mathcal{D}^{k+1}}^p(\mathcal{E}_{\mathcal{D}^{k+1}})$ is well-defined, so as to obtain the sequence (2.7). The exactness follows from that on the atlas, which is standard (cf. [Fuj09, p. 40]).

Finally, using the standard perfect pairing

$$\Omega_{\mathcal{X}}^p(\log \mathcal{D}) \times \Omega_{\mathcal{X}}^{n-p}(\log \mathcal{D}) \rightarrow \omega_{\mathcal{X}}(\mathcal{D}), \quad (2.9)$$

we obtain (2.8) from (2.7). \square

Lemma 2.6. *Let*

$$F^\bullet = (\cdots \rightarrow 0 \rightarrow F^0 \rightarrow F^1 \rightarrow \cdots) \quad (2.10)$$

be a complex of sheaves on a Deligne-Mumford stack \mathcal{X} . Then there exists the following spectral sequence.

$$E_1^{p,q} = H^q(\mathcal{X}, F^p) \Rightarrow E^{p+q} = \mathbb{H}^{p+q}(\mathcal{X}, F^\bullet) \quad (2.11)$$

Proof. This is standard. For example, see [PS08, Lemma-Definition A.46]. \square

We need the following variant of [DI87, Theorem 2.1] for a Deligne-Mumford stack.

Lemma 2.7. *Let $(\mathcal{X}, \mathcal{D})$ be a normal crossing pair of Deligne-Mumford stack over a perfect field \mathbf{k} of characteristic $p > 0$. Let $S := \operatorname{Spec} \mathbf{k}$ and $F = F_{X/S}: \mathcal{X} \rightarrow \mathcal{X}' := \mathcal{X} \times_{S, F_S} S$ be the relative Frobenius morphism as explained in [Sat12, Section 1.1]. Let $W_2(\mathbf{k})$ be the ring of truncated Witt vectors and $\tilde{S} := \operatorname{Spec} W_2(\mathbf{k})$.*

(i) There is a unique isomorphism of graded $\mathcal{O}_{\mathcal{X}'}$ -algebras

$$C^{-1}: \bigoplus_{i \geq 0} \Omega_{\mathcal{X}'/S}^i(\log \mathcal{D}') \rightarrow \bigoplus_{i \geq 0} \mathcal{H}^i(F_* \Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{D}))$$

such that $C^{-1}(d(x \otimes 1)) = x^{p-1}dx$.

(ii) Assume that there exists a lift $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}})$ of $(\mathcal{X}, \mathcal{D})$ over \tilde{S} . Then there is an associated isomorphism

$$\varphi: \bigoplus_{i < p} \Omega_{\mathcal{X}'/S}^i(\log \mathcal{D}') \rightarrow \tau_{< p} F_* \Omega_{\mathcal{X}/S}^\bullet(\log \mathcal{D})$$

in the derived category of $\mathcal{O}_{\mathcal{X}'}$ -modules such that $\mathcal{H}^i(\varphi) = C^{-1}$ for $i < p$.

Proof. The first claim is the logarithmic version of [Sat12, Corollary 2.2]. The original proof literally works if we replace the Cartier isomorphism with its logarithmic version due to Katz [Kat70, (7.2.4)].

The second claim for schemes is in [DI87, 4.2.3]. With a little more care, as we explain next, the proof works for Deligne-Mumford stacks as well. Below we use the symbols of [Sat12, Section 1.1].

Consider an étale cover $\mathcal{U} = \{U_i\}$ of the stack \mathcal{X} such that U_i are affine. Let $D_i := \mathcal{D}|_{U_i}$ and $(\tilde{U}_i, \tilde{D}_i)$ a lift of (U_i, D_i) over \tilde{S} induced by $(\tilde{\mathcal{X}}, \tilde{\mathcal{D}})$. Since U'_i are affine, as explained in [EV92, Example 10.3], we obtain a lift $\tilde{F}_i := \tilde{F}_{U_i/S}$ of $F_{U_i/S}$ satisfying $\tilde{F}_i^{-1}(\tilde{D}'_i) = p\tilde{D}_i$. Thus we obtain the morphisms $f_i: \Omega_{U'_i/S}^1(\log D'_i) \rightarrow (F_{U_i/S})_* \Omega_{U_i/S}^1(\log D_i)$. Now, as pointed out in [Sat12, Remark 1.4], the relative Frobenii commutes with étale morphisms. This ensures that the homotopy maps “ $h_{ij} = \tilde{F}_j^* - \tilde{F}_i^*$ ” relating f_j and f_i are well defined. The rest of the construction of φ works without change. \square

Lemma 2.8. *Let \mathcal{X} be a smooth proper Deligne-Mumford stack over \mathbb{C} and $\mathcal{D} \subset \mathcal{X}$ a normal crossing divisor. Then the spectral sequence*

$$E_1^{p,q} := H^q(\mathcal{X}, \Omega_{\mathcal{X}}^p(\log \mathcal{D})) \Rightarrow \mathbb{H}^{p+q}(\mathcal{X}, \Omega_{\mathcal{X}}^\bullet(\log \mathcal{D})) \simeq H^{p+q}((\mathcal{X} \setminus \mathcal{D})^{\text{an}}, \mathbb{C})$$

degenerates at E_1 . As a consequence, we have an isomorphism

$$H^i((\mathcal{X} \setminus \mathcal{D})^{\text{an}}, \mathbb{C}) \cong \bigoplus_{q+p=i} H^q(\mathcal{X}, \Omega_{\mathcal{X}}^p(\log \mathcal{D})) \quad (2.12)$$

for any $i \geq 0$.

Proof. When \mathcal{X} is a scheme, this is proved in [DI87, Corollaire 4.2.4]. We can generalize this to the case where \mathcal{X} is a Deligne-Mumford stack by Lemma 2.7 and arguments as in [Sat12, Corollary 1.7] and the proof of [Sat12, Lemma 1.9]. \square

As an application we obtain the stacky and logarithmic generalization of the Kodaira-Akizuki-Nakano vanishing theorem.

Definition-Proposition 2.9. Let \mathcal{X} be a Deligne-Mumford stack which admits a coarse moduli scheme $p: \mathcal{X} \rightarrow X$. Then for any line bundle \mathcal{L} on \mathcal{X} , there exists a positive integer N such that $\mathcal{L}^{\otimes N} \cong p^*L$ holds for some line bundle L on X .

We say that \mathcal{L} has a property \mathcal{P} if the descent L as above has the property \mathcal{P} ; here \mathcal{P} is a property of line bundles on schemes which depends only on the isomorphism classes of \mathbb{Q} -line bundles, such as ampleness.

Proposition 2.10. *Let \mathcal{X} be a smooth proper Deligne-Mumford stack of dimension d over a perfect field \mathbf{k} of characteristic $p \geq 0$, \mathcal{D} a normal crossing divisor on \mathcal{X} , and \mathcal{L} an ample line bundle on \mathcal{X} . Then we have*

$$H^i(\mathcal{X}, \Omega_{\mathcal{X}}^j(\log \mathcal{D})(-\mathcal{D}) \otimes \mathcal{L}) = 0 \quad (2.13)$$

if $i + j > \mathbf{d}(d, p)$, where

$$\mathbf{d}(d, p) := \begin{cases} \sup(d, 2d - p) & p > 0, \\ d & p = 0. \end{cases} \quad (2.14)$$

Proof.

Step 1. Let us first consider the case $\mathcal{D} = \emptyset$. Since we have Lemma 2.7, we can prove the statement by the same arguments as in the proof of [DI87, Corollaire 2.8 (i)] when $p > 0$.

We can also prove the statement for $p = 0$ by the same arguments as in the proof of [DI87, Corollaire 2.11]. Note that there exists an integral domain A of finite type over \mathbb{Z} , an injective homomorphism $A \hookrightarrow \mathbf{k}$, and a smooth proper Deligne-Mumford stack \mathcal{Y} over A which pulls back to \mathcal{X} over \mathbf{k} as explained in [Sat12, Corollary 1.7].

Step 2. Applying Lemma 2.6 to the exact sequence (2.7) tensored with \mathcal{L} , we obtain the spectral sequence

$$E_1^{s,t} = H^t(\mathcal{D}^s, \Omega_{\mathcal{D}^s}^j(\mathcal{E}_{\mathcal{D}^s}) \otimes (\varphi^s)^* \mathcal{L}) \Rightarrow E^{s+t} = H^{s+t}(\mathcal{X}, \Omega_{\mathcal{X}}^j(\log \mathcal{D})(-\mathcal{D}) \otimes \mathcal{L}).$$

Then for each (s, t) such that $s + t = i$, we have $E_1^{s,t} = 0$. This follows from Step 1, since $\mathcal{E}_{\mathcal{D}^s} \otimes (\varphi^s)^* \mathcal{L}$ is ample and $j + t = j + i - s > \mathbf{d}(d, p) - s \geq \mathbf{d}(\dim \mathcal{D}^s, p)$. Thus we obtain $E^i = 0$ to conclude the proof. □

2.2. Recap of $\overline{\mathcal{M}}_{g,n}$. We recall some standard facts about the moduli of stable pointed curves. For statements without proofs, see [Hac08] and references therein.

Given a pair of integers (g, n) satisfying the three conditions

$$n \geq 0, \quad g \geq 0, \quad 2g - 2 + n > 0, \quad (2.15)$$

we have the category $\overline{\mathcal{M}}_{g,n}$ of stable n -pointed genus g curves fibered in groupoids over the category $(\mathbf{Sch}/\mathbf{k})$. It is a smooth proper Deligne-Mumford stack of dimension $3g - 3 + n$. There exists the universal curve $\pi: \mathcal{U}_{g,n} \rightarrow \overline{\mathcal{M}}_{g,n}$, which itself is a smooth proper Deligne-Mumford stack, and for $i = 1, 2, \dots, n$ we denote by $\sigma_i: \overline{\mathcal{M}}_{g,n} \rightarrow \mathcal{U}_{g,n}$ the section of π representing the i -th marked point. With these notations, the ψ -line bundles on $\overline{\mathcal{M}}_{g,n}$ are defined as $\psi_i := (\sigma_i)^* \omega_\pi$ for $i = 1, 2, \dots, n$.

Lemma 2.11. *Suppose $n > 0$. Then for any $i = 1, 2, \dots, n$, the line bundles ψ_i are nef and big.*

Proof. For $i = n$, the assertion follows from [Hac08, Theorem 3.2 and Lemma 4.3] (see also [Kee99, Section 4]). For other i , use the fact that the symmetry group \mathfrak{S}_n acts on $\overline{\mathcal{M}}_{g,n}$ in such a way that the transposition (i, n) sends ψ_i to ψ_n . □

There exists a closed substack $\mathcal{B} \subset \overline{\mathcal{M}}_{g,n}$ representing singular stable pointed curves. It is a reduced normal crossing divisor. The open substack of smooth pointed stable curves is denoted by $\mathcal{M}_{g,n} := \overline{\mathcal{M}}_{g,n} \setminus \mathcal{B}$.

The coarse moduli space $p: \overline{\mathcal{M}}_{g,n} \rightarrow \overline{M}_{g,n}$ is a projective variety. When $g = 0$, the morphism p is an isomorphism. The following positivity is an essential ingredient of our proof of Theorem 3.2.

Proposition 2.12. *For any (g, n) satisfying (2.15), the log canonical bundle $\omega_{\overline{\mathcal{M}}_{g,n}}(\mathcal{B})$ is ample.*

Proof. This is a special case of [Fed11, Theorem 4.1]. \square

We also use the inductive nature of the boundary divisors.

Lemma 2.13. *Any connected component \mathcal{B}' of the normalization $\mathcal{B}^1 \rightarrow \mathcal{B}$ is smooth, and admits a finite étale morphism $\mathcal{C}' \rightarrow \mathcal{B}'$ such that \mathcal{C}' is isomorphic to either*

- (1) $\overline{\mathcal{M}}_{g',n'} \times \overline{\mathcal{M}}_{g'',n''}$, where $g' + g'' = g$ and $n' + n'' = n + 2$, or
- (2) $\overline{\mathcal{M}}_{g-1,n+2}$.

Moreover, in the latter case, the morphism is a two-to-one Galois covering.

Proof. See [Hac08, p. 813]. \square

3. PROOF OF THEOREM 1.1

Recall that the dimension of cohomology space is invariant under base field extensions. Since the moduli stacks of pointed stable curves are already defined over \mathbb{Q} , we can and will work over \mathbb{C} throughout this section.

The following two statements are shown in this section.

Theorem 3.1. *The vanishing*

$$H^i(\overline{\mathcal{M}}_{g,n}, \mathcal{O}_{\overline{\mathcal{M}}_{g,n}}) = 0 \quad (3.1)$$

holds for $i = 1, 2$.

Theorem 3.2. *The vanishing*

$$H^0(\overline{\mathcal{M}}_{g,n}, \wedge^2 \Theta_{\overline{\mathcal{M}}_{g,n}}) = 0 \quad (3.2)$$

holds if $(g, n) \neq (0, 5), (1, 2)$.

Theorem 3.2 for the exceptional cases $(g, n) = (0, 5)$ and $(1, 2)$ will be treated in Section 3.3.

3.1. Proof of Theorem 3.2.

Lemma 3.3. *Let \mathcal{L} be a nef line bundle on $\overline{\mathcal{M}}_{g,n}$. Then*

$$H^q(\overline{\mathcal{M}}_{g,n}, \wedge^p \Theta_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \otimes \mathcal{L}^{-1}) = 0 \quad (3.3)$$

holds for any p, q with $p > q$.

Proof. The Serre duality tells us

$$H^q(\overline{\mathcal{M}}_{g,n}, \wedge^p \Theta_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \otimes \mathcal{L}^{-1})^\vee \cong H^{d-q}(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^p(\log \mathcal{B})(-\mathcal{B}) \otimes (\omega_{\overline{\mathcal{M}}_{g,n}}(\mathcal{B}) \otimes \mathcal{L})). \quad (3.4)$$

Since $\omega_{\overline{\mathcal{M}}_{g,n}}(\mathcal{B}) \otimes \mathcal{L}$ is always ample by Proposition 2.12, by applying Proposition 2.10, we see that the cohomology space is trivial when $(d - q) + p > d \iff p > q$. \square

Proof of Theorem 3.2. Let us fix the value of (p, q) . Applying Lemma 2.6 to (2.8), we obtain the following spectral sequence.

$$E_1^{s,t} = H^t \left(\mathcal{B}^s, \wedge^{p-s} \Theta_{\mathcal{B}^s}(\mathcal{E}_{\mathcal{B}^s}) \otimes \det(\mathcal{N}_{\mathcal{B}^s/\overline{\mathcal{M}}_{g,n}}) \right) \Rightarrow H^{s+t} \left(\overline{\mathcal{M}}_{g,n}, \wedge^p \Theta_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \right) \quad (3.5)$$

Note that $E_1^{0,q} = H^q(\overline{\mathcal{M}}_{g,n}, \wedge^p \Theta_{\overline{\mathcal{M}}_{g,n}})$.

By Lemma 3.3, if we have the inequality $p > q$, we immediately see $E^q = 0$. If moreover we have the vanishings $E_1^{1,q} = E_1^{2,q-1} = E_1^{3,q-2} = \dots = E_1^{q+1,0} = 0$, then we obtain the vanishing of $E_1^{0,q}$.

In the rest we use this strategy to deal with the case $(p, q) = (2, 0)$, namely Theorem 3.2, under the assumption $\dim \overline{\mathcal{M}}_{g,n} \geq 3$. In this case, since we want to show the vanishing of $E_1^{0,0}$, all we have to show is the vanishing of

$$E_1^{1,0} = H^0(\mathcal{B}^1, \Theta_{\mathcal{B}^1}(\mathcal{E}_{\mathcal{B}^1}) \otimes \mathcal{N}_{\mathcal{B}^1/\overline{\mathcal{M}}_{g,n}}) = H^0(\mathcal{B}^1, \Theta_{\mathcal{B}^1} \otimes \mathcal{N}_{\mathcal{B}^1/\overline{\mathcal{M}}_{g,n}}). \quad (3.6)$$

Note that $\mathcal{E}_{\mathcal{B}^1} \simeq \mathcal{O}_{\mathcal{B}^1}$ as explained in the end of Definition-Proposition 2.4.

Let \mathcal{B}' be a connected component of \mathcal{B}^1 , and consider the finite étale Galois cover $\pi: \mathcal{C}' \twoheadrightarrow \mathcal{B}'$ as in Lemma 2.13. Since we have the inclusion

$$\pi^*: H^0(\mathcal{B}', \Theta_{\mathcal{B}'} \otimes \mathcal{N}_{\mathcal{B}'/\overline{\mathcal{M}}_{g,n}}) \hookrightarrow H^0(\mathcal{C}', \Theta_{\mathcal{C}'} \otimes \pi^* \mathcal{N}_{\mathcal{B}'/\overline{\mathcal{M}}_{g,n}}), \quad (3.7)$$

it is enough to show the vanishing of the RHS.

If $\mathcal{C}' \cong \overline{\mathcal{M}}' \times \overline{\mathcal{M}}''$, then by [Hac08, Lemma 4.2] and the Künneth formula, the RHS is isomorphic to

$$\left(H^0(\overline{\mathcal{M}}', \Theta_{\overline{\mathcal{M}}'} \otimes \psi_{n+1}^\vee) \otimes H^0(\overline{\mathcal{M}}'', \psi_{n+2}^\vee) \right) \oplus \left(H^0(\overline{\mathcal{M}}'', \Theta_{\overline{\mathcal{M}}''} \otimes \psi_{n+2}^\vee) \otimes H^0(\overline{\mathcal{M}}', \psi_{n+1}^\vee) \right). \quad (3.8)$$

Since ψ_i are nef and big by Lemma 2.11, (3.8) is trivial when $\dim \overline{\mathcal{M}}'$ and $\dim \overline{\mathcal{M}}''$ are both positive. Otherwise we can use Proposition 3.4 below.

If $\mathcal{C}' \cong \overline{\mathcal{M}}_{g-1,n+2}$, again by [Hac08, Lemma 4.2], the RHS is isomorphic to

$$H^0(\overline{\mathcal{M}}_{g-1,n+2}, \Theta_{\overline{\mathcal{M}}_{g-1,n+2}} \otimes \psi_{n+1}^\vee \otimes \psi_{n+2}^\vee). \quad (3.9)$$

Since $\psi_{n+1} \otimes \psi_{n+2}$ is nef and big by Lemma 2.11, we can use Proposition 3.4 again to conclude the proof. \square

Proposition 3.4. *For any anti-nef line bundle \mathcal{L} , $H^0(\overline{\mathcal{M}}_{g,n}, \Theta_{\overline{\mathcal{M}}_{g,n}} \otimes \mathcal{L}) = 0$ holds if $\dim \overline{\mathcal{M}}_{g,n} \geq 2$.*

Proof. By applying $\otimes \mathcal{L}$ to (2.8) with $p = 1$, we obtain the exact sequence

$$H^0(\overline{\mathcal{M}}_{g,n}, \Theta_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \otimes \mathcal{L}) \rightarrow H^0(\overline{\mathcal{M}}_{g,n}, \Theta_{\overline{\mathcal{M}}_{g,n}} \otimes \mathcal{L}) \rightarrow H^0(\mathcal{B}^1, \mathcal{L} \otimes \mathcal{E}_{\mathcal{B}^1} \otimes \mathcal{N}_{\mathcal{B}^1/\overline{\mathcal{M}}_{g,n}}). \quad (3.10)$$

Since $\mathcal{L} \otimes \mathcal{E}_{\mathcal{B}^1} \otimes \mathcal{N}_{\mathcal{B}^1/\overline{\mathcal{M}}_{g,n}}$ is anti-nef and big on the finite étale cover \mathcal{C}' of Lemma 2.13 and $\dim \mathcal{B}^1 \geq 1$, the third term has to be trivial. Finally, the vanishing of the first term follows from Lemma 3.3. \square

3.1.1. *Interlude.* Since Proposition 2.10 is valid over arbitrary perfect fields and the moduli stacks $\overline{\mathcal{M}}_{g,n}$ are defined over prime fields, one can easily check that Theorem 3.2 is generalized to positive characteristics as follows.

Theorem 3.5. *Let \mathbf{k} be a field of characteristic $p > 0$, and suppose (g, n) satisfies the inequalities $3 \leq 3g - 3 + n \leq p$. Then $H^0(\overline{\mathcal{M}}_{g,n}, \wedge^2 \Theta_{\overline{\mathcal{M}}_{g,n}}) = 0$.*

Proof. The inequality $3g - 3 + n \leq p$ ensures the equality $\mathbf{d}(\dim \mathcal{B}^i, p) = \dim \mathcal{B}^i$ for any strata \mathcal{B}^i of $\overline{\mathcal{M}}_{g,n}$. Therefore the arguments in the case of characteristic zero works without change. \square

3.2. Proof of Theorem 3.1. Here we prove Theorem 3.1. The case $g = 0$ is trivial, since $\overline{\mathcal{M}}_{0,n}$ are rational. When $g = 1$ and $n \leq 2$, the assertion follows from the rationality of $\overline{\mathcal{M}}_{1,1}$ and $\overline{\mathcal{M}}_{1,2}$ (see [Mas14, Theorem 2.3]). In the rest of this subsection we assume

$$g \geq 2 \quad \text{or} \quad [g = 1 \text{ and } n > 2], \quad (3.11)$$

so that we always have $d := \dim \overline{\mathcal{M}}_{g,n} = 3g - 3 + n \geq 3$. By Lemma 2.3, the claim of Theorem 3.1 is equivalent to the vanishing $H^0(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^i) = 0$ for $i = 1, 2$. Below is the key ingredient of the proof.

Fact 3.6. *Under our assumption (3.11), $H^i(\mathcal{M}_{g,n}, \mathbb{Q}) = 0$ holds for $i = 2d - 1, 2d - 2$.*

Proof. It was proven by Harer (see [ACG11, Chapter 19, Theorem (2.2)]) that the vanishing $H_i(M_{g,n}, \mathbb{Q}) = 0$ holds when

$$i > \begin{cases} n - 3 & (g = 0) \\ 4g - 5 & (g > 0, n = 0) \\ 4g - 4 + n & (g > 0, n > 0). \end{cases} \quad (3.12)$$

We can easily check that (3.12) holds for $i = 2d - 2$ and hence also for $i = 2d - 1$, under our assumption (3.11). Finally we have $H_i(M_{g,n}, \mathbb{Q}) \cong (H^i(M_{g,n}, \mathbb{Q}))^\vee$ (see [MS74, Theorem A1]) and $H^i(M_{g,n}, \mathbb{Q}) \cong H^i(\mathcal{M}_{g,n}, \mathbb{Q})$ (see [Beh04, Proposition 36]). \square

Proof of Theorem 3.1. Let us deal with the case $i = 1$ first. Consider the exact sequence

$$H^0(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^1(\log \mathcal{B})(-\mathcal{B})) \rightarrow H^0(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^1) \rightarrow H^0(\mathcal{B}^1, \Omega_{\mathcal{B}^1}^1) \quad (3.13)$$

obtained from (2.7). We check the vanishing of the first term of (3.13). Using the Serre duality and the perfect pairing (2.9), we obtain

$$\begin{aligned} H^0(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^i(\log \mathcal{B})(-\mathcal{B})) &\cong H^0(\overline{\mathcal{M}}_{g,n}, \wedge^{d-i} \Theta_{\overline{\mathcal{M}}_{g,n}}(-\log \mathcal{B}) \otimes \omega_{\overline{\mathcal{M}}_{g,n}}) \\ &\cong H^d(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^{d-i}(\log \mathcal{B}))^\vee \end{aligned} \quad (3.14)$$

for any i . The dual of the last term is, by Lemma 2.8, a direct summand of $H^{2d-i}(\mathcal{M}_{g,n}, \mathbb{Q})$. As we saw in Fact 3.6, this is always trivial when $i = 1$.

Next we show the vanishing of the third term of (3.13) by an induction on $\dim \overline{\mathcal{M}}_{g,n}$, starting with the case when $\dim \overline{\mathcal{M}}_{g,n} = 3$. In the initial case we have $\dim \mathcal{B}' = 2$ and hence can check the assertion by hand, since then \mathcal{B}' is always rational. For general cases, take any connected component $\mathcal{B}' \subset \mathcal{B}^1$ and the finite étale cover $\pi: \mathcal{C}' \twoheadrightarrow \mathcal{B}'$ of Lemma 2.13. Since $\pi^* \Omega_{\mathcal{B}'}^1 \cong \Omega_{\mathcal{C}'}^1$, it is enough to show $H^0(\mathcal{C}', \Omega_{\mathcal{C}'}^1) = 0$. Suppose $\mathcal{C}' \cong \overline{\mathcal{M}}_{g-1, n+2}$. Then we can use the induction hypothesis if $(g-1, n+2)$ still satisfies (3.11), and otherwise we

already checked the assertion in the beginning of this subsection. If $\mathcal{C}' \cong \overline{\mathcal{M}}_{g',n'} \times \overline{\mathcal{M}}_{g'',n''}$, where $g' + g'' = g$ and $n' + n'' = n + 2$, then by using the Künneth formula we can check the assertion by the similar arguments. Thus we conclude the proof for the case $i = 1$.

Finally we consider the case $i = 2$. The arguments are essentially the same as in the case $i = 1$. Consider the exact sequence

$$H^0(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^2(\log \mathcal{B})(-\mathcal{B})) \rightarrow H^0(\overline{\mathcal{M}}_{g,n}, \Omega_{\overline{\mathcal{M}}_{g,n}}^2) \rightarrow H^0(\mathcal{B}^1, \Omega_{\mathcal{B}^1}^2) \quad (3.15)$$

obtained from (2.7). The triviality of the first term follows from the same arguments. To see the vanishing of the third term by an induction, we consider the finite étale cover $\mathcal{C}' \rightarrow \mathcal{B}'$ as before and check the vanishing of $H^0(\mathcal{C}', \Omega_{\mathcal{C}'}^2)$. As a new ingredient, when $\mathcal{C}' \cong \overline{\mathcal{M}}' \times \overline{\mathcal{M}}''$, we use the Künneth formula and need the result for $i = 1$ to show the vanishing of the term

$$H^0(\mathcal{C}', \Omega_{\overline{\mathcal{M}}'}^1 \boxtimes \Omega_{\overline{\mathcal{M}}''}^1) \cong H^0(\overline{\mathcal{M}}', \Omega_{\overline{\mathcal{M}}'}^1) \otimes H^0(\overline{\mathcal{M}}'', \Omega_{\overline{\mathcal{M}}''}^1). \quad (3.16)$$

Thus we conclude the proof. \square

3.3. Settling exceptional cases: $\overline{\mathcal{M}}_{0,5}$ and $\overline{\mathcal{M}}_{1,2}$. Recall first that $\overline{\mathcal{M}}_{0,5}$ is isomorphic to the blowup of \mathbb{P}^2 in a general set of four points. This immediately tells us

Lemma 3.7. $\dim H^0(\overline{\mathcal{M}}_{0,5}, \wedge^2 \Theta_{\overline{\mathcal{M}}_{0,5}}) = 6$.

We next prove

Lemma 3.8. $\mathrm{HH}^3(\overline{\mathcal{M}}_{0,5}) = 0$.

Proof. By the HKR isomorphism, we see that $\mathrm{HH}^3(\overline{\mathcal{M}}_{0,5})$ is isomorphic to the direct sum

$$H^3(\overline{\mathcal{M}}_{0,5}, \mathcal{O}_{\overline{\mathcal{M}}_{0,5}}) \oplus H^2(\overline{\mathcal{M}}_{0,5}, \Theta_{\overline{\mathcal{M}}_{0,5}}) \oplus H^1(\overline{\mathcal{M}}_{0,5}, \wedge^2 \Theta_{\overline{\mathcal{M}}_{0,5}}) \oplus H^0(\overline{\mathcal{M}}_{0,5}, \wedge^3 \Theta_{\overline{\mathcal{M}}_{0,5}}). \quad (3.17)$$

Since $\dim \overline{\mathcal{M}}_{0,5} = 2$, we see $H^3(\overline{\mathcal{M}}_{0,5}, \mathcal{O}_{\overline{\mathcal{M}}_{0,5}}) = 0$ and $\wedge^3 \Theta_{\overline{\mathcal{M}}_{0,5}} = 0$. Moreover, using the canonical isomorphism $\Theta_{\overline{\mathcal{M}}_{0,5}} \cong \Omega_{\overline{\mathcal{M}}_{0,5}}^1 \otimes \omega_{\overline{\mathcal{M}}_{0,5}}^{-1}$ and applying Proposition 2.10 to the ample line bundle $\mathcal{L} = \omega_{\overline{\mathcal{M}}_{0,5}}^{-1}$, we can show the vanishing of the two middle terms of (3.17). \square

Recall next from [Mas14, Section 2] the description of $\overline{\mathcal{M}}_{1,2}$ as a global quotient stack. Consider the smooth variety

$$X := Z(zy^2 - x^3 - axz^2 - bz^3) \subset \mathbb{A}_0^3 \times \mathbb{A}_0^2. \quad (3.18)$$

Here $\mathbb{A}_0^* := \mathbb{A}^* \setminus \{0\}$, (x, y, z) are the coordinates of \mathbb{A}^3 , and (a, b) are the coordinates of \mathbb{A}^2 . The torus $G = \mathbb{G}_m^2$ acts on $\mathbb{A}_0^3 \times \mathbb{A}_0^2$ with the weights in the table below.

	x	y	z	a	b
ξ	1	1	1	0	0
λ	2	3	0	4	6

Here (ξ, λ) are the coordinates of G . Note that the defining equation

$$f := zy^2 - x^3 - axz^2 - bz^3 \quad (3.19)$$

is homogeneous of weight $(3, 6)$. Then we have a natural isomorphism of stacks

$$[X/G] \xrightarrow{\sim} \overline{\mathcal{M}}_{1,2}. \quad (3.20)$$

The closure $\overline{X} \subset \mathbb{A}^5$ is the normal hypersurface defined by the equation f . As explained in [CPR00, Proof of Lemma 3.5], the divisor class group of \overline{X} is trivial. Therefore $\mathcal{O}_{\overline{X}}$ is “the Cox ring” of the Deligne-Mumford stack $[X/G]$, and under the canonical equivalence

$$\pi^*: \mathrm{Coh}[X/G] \xrightarrow{\sim} \mathrm{Coh}^G X, \quad (3.21)$$

we see the coincidence (as linearized line bundles) $\pi^*\omega_{[X/G]} \cong \omega_X$.

Proposition 3.9. $H^0(\overline{\mathcal{M}}_{1,2}, \wedge^2 \Theta_{\overline{\mathcal{M}}_{1,2}}) = 0$.

Proof. Set $U := \mathbb{A}_0^3 \times \mathbb{A}_0^2$ for simplicity. By the adjunction formula, we see

$$\omega_{[X/G]} \cong \omega_{[U/G]}|_{[X/G]} \otimes \mathcal{O}_{[X/G]}([X/G]). \quad (3.22)$$

Under the equivalence (3.21), it is translated into

$$\begin{aligned} \pi^*\omega_{[X/G]} &\cong \omega_X \cong \omega_U|_X \otimes \mathcal{O}_X(X) \cong (\mathcal{O}_X \cdot dx \wedge dy \wedge dz \wedge da \wedge db) \otimes (\mathcal{O}_X \cdot f)^\vee \\ &\cong \mathcal{O}_X(-3, -15) \otimes \mathcal{O}_X(3, 6) \cong \mathcal{O}_X(0, -9). \end{aligned} \quad (3.23)$$

Therefore we see

$$H^0([X/G], \wedge^2 \Theta_{[X/G]}) \cong \mathrm{Hom}_X^G(\mathcal{O}_X, \mathcal{O}_X(0, 9)) \cong (\mathbf{k}[x, y, z, a, b]/(f))_{(0,9)} = 0. \quad (3.24)$$

The last equality is due to the absence of monomials of bidegree $(0, 9)$. \square

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